B. Math. Hons. II Year
II Semester 2001-2002
Midsemestral Exam : Algebra IV
Date: 25.02.2002 Time: 9.45-12.45 Instructor: B.Sury

Answer all questions. All questions carry equal marks.

1. Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be irreducible, finite-dimensional, complex representations of a finite group G. Determine the linear transformations $T : V \to W$ such that
$$T \circ \rho(g) = \sigma(g) \circ T \ \forall \ g \in G.$$

2. Let $\rho : G \to GL(V)$ be a complex, irreducible representation of a finite group G. Prove that, upto scalars, there is a unique Hermitian inner product on V which is $\rho(G)$-invariant.

3. State Schur's first orthogonality relations and use it to prove that the character of a complex, finite-dimensional representation of a finite group determines the representation uniquely.

4. Consider the following character table of a finite group G:

	w	x	y	z
χ_{ρ_1}	1	1	1	1
χ_{ρ_2}	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$	1
χ_{ρ_3}	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$	1
χ_{ρ_4}	3	0	0	-1

Find: (a) $O(G)$, (b) which one of w, x, y, z is central, (c) cardinalities of the conjugacy classes and (d) Dimensions of the irreducible representations.

5. Prove that over a field K, it is impossible to find an injective, irreducible representation
$$\rho : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to GL(V).$$

6. For any left R-module M over any ring R and, $\forall\ n \geq 1$, show that there is a ring isomorphism

$$End_R M^n \cong M_n(End_R M).$$

7. Let D be a division ring and let V be a left D-vector space of dimension 1. Show that $End_D V \cong D^{op}$ as rings, where D^{op} is the opposite ring of D (i.e., has the same underlying additive group as D and with multiplication $a * b$ defined as $a * b = ba$ where the right side denotes multiplication in D.)

8. Let M be any simple left R-module and let $x \in M$. Prove that, for any $\phi \in End_{End_R M} M, \exists\ r \in R$ such that $\phi(x) = rx$.

9. Let Char.$K = p > 0$ and suppose G is a finite group where $p/O(G)$. Show that the element $\sum\limits_{g \in G} g \in K[G]$ satisfies $\left(\sum\limits_{g \in G} g\right)^2 = 0$. Hence, show that $K[G]$ is not semisimple.